PERIOD-II

M10CH3

TOPIC

3 Algebraic Expression

3.1 ALGEBRAIC STATEMENT

In algebra, a statement is basically sentences which are *either* true *or* false. Those mathematical statements which may consist of words and symbols are called algebraic statements.

For example:

(*i*) The square root of a number is 2 is an algebraic mathematical statement.

(*ii*) 3 times the difference of a number and 8

(*iii*) 5 times the sum of a number and 6

(*iv*) The sum of 5 times a number and 6

(*v*) 8 times the difference of twice a number and 6.

3.2 NUMERICAL STATEMENTS

A numerical statement is a *mathematical phrase* or *problem* made up *entirely of numerical characters.* All levels of numerical statements present themselves daily in our lives.

There are statements which have a definite response; *i.e.*, these are *either* 'true' *or* 'false'.

For example:

(*a*) 7 nines is 64 (*b*) $2 + 3 = 5$

(*c*) $4 \times 6 = 10$ (*d*) $20 \div 5 = 4$

(*e*) $\sqrt{49} = 7$ (*f*) $2^4 = 16$

Here the statements (*b*), (*d*), (*e*) and (*f*) are true.

EXERCISE 3.1

1. Write down the algebraic statements from the following algebraic expressions:

(*a*) $3x$ (*b*) $-5y$ (*c*) $-7t + q$ (*d*) $6 - x^2$

(*e*) $100p - 100q$ (*f*) $x - y$

2. Indicate if the following statements are true or false:

(*a*) $2 + 6 = 8$ (*b*) 9 divides 55 (*c*) 4 is a prime number.

(*d*) $7 - 3 = 4$ (*e*) $8^2 = 65$ (*f*) $\sqrt{100} = 10$

3.3 FORMING AN ALGEBRAIC EXPRESSION

Consider an expression: $5x + 3$

1. x is called the literal *or* the variable as its value changes.
2. 3 is called a constant as its value does not change.
3. 5 is called the numerical coefficient of x.
4. $5x + 3$ is called an algebraic expression.

To make this expression, we say that "5 times the number x is added to 3", *i.e.*, $5x + 3$.

Example 1. *Make algebraic expressions of the following statements:*

(*i*) *The sum of 9 and a number x.*

(*ii*) *6 increased by a number y.*

(*iii*) *The product of a number x and 7.*

(*iv*) *Sum of x and y added to 3 times x.*

(*v*) *Product of x and y added to 8.*

Solution.

(*i*) $9 + x$ (*ii*) $6 + y$ (*iii*) $7x$

(*iv*) Sum of x and y gives $x + y$.

Sum of x and y added to 3 times x gives $(x + y) + 3x$.

(*v*) Product of x and y added to 8 gives $xy + 8$.

EXERCISE 3.2

1. Pick the variables and constants in each expression.

(*a*) $7x + 3$ (*b*) $x - y$ (*c*) $6 - x^2$ (*d*) $3x^2 - 2x + y$

(*e*) $8y + xy - 6$

2. Write down the coefficient of each literal:

(*a*) $6x + 2$ (*b*) $-3x - 4$ (*c*) $-4x + 2y$ (*d*) $2x + 3y + 4z$

(*e*) $100p - 100q$

3. Write down the literal coefficient:

(*a*) $3x$ (*b*) $-5y$ (*c*) $-7t$ (*d*) $6xy$

(*e*) $-25x^2 y^2 z^2$ (*f*) $61pq$

4. Form algebraic expressions of the following statements.

(*a*) The sum of 3 and a number x.

(*b*) Six less than a number t.

(*c*) The product of –5 and a number x.

(*d*) A number y divided by 16.

(*e*) 3 times the number x added to 4 times the number y.

3.4(A) EVALUATING ALGEBRAIC EXPRESSION

In order to evaluate an algebraic expression, you must know the exact values for each variable. Then you will simply substitute and evaluate using the order of operations.

Example 2. *Evaluate $x + 7$ when*

(*i*) $x = 3$ (*ii*) $x = 12$.

Solution. (*i*) To evaluate, substitute 3 for x in the expression, and then simplify

	$x + 7$
Substitute	**3** + 7
Add	10

Thus, 10 is the final answer.

(*ii*)	$x + 7$
Substitute	**12** + 7
Add	19

Notice that we got different results for part (*i*) and part (*ii*) even though we started with the same expression. This is because the values used for *x* were different. When we evaluate an expression, the value varies depending on the value used for the variable.

Example 3. *Evaluate the expression* $a + \frac{3 + b^3}{2} - c$ *when* $a = 4$, $b = 3$ *and* $c = 8$.

Solution. We have

$a + (3 + b^3)/2 - c$	
↓	
$4 + (3 + 3^3)/2 - 8$	Substitute the given values for each values
↓	
$4 + (3 + 27)/2 - 8$	We must start the parenthesis and evaluate the power (exponent) ($3^3 = 27$)
↓	
$4 + 30/2 - 8$	Evaluate the parenthesis ($3 + 27 = 30$)
↓	
$4 + 15 - 8$	Evaluate the division ($30 \div 2 = 15$)
↓	
$19 - 8$	Now, addition comes first ($4 + 15 = 19$)
11	Finally, subtraction ($19 - 8 = 11$)

Thus, 11 is the final answer.

3.4(B) RELATIONS BETWEEN TWO ALGEBRAIC EXPRESSIONS

Addition and Subtraction of Algebraic Expressions

To add or subtract algebraic expressions:

(*i*) We write one expression below the other such that the like terms are written one below the other. Then we add or subtract each term. In subtraction, we change the sign of each term of the expression that is to be subtracted. This is called the *Column Method.*

(*ii*) In the horizontal method, for addition, like terms are grouped together and then combined. For subtraction, the sign of each term of the expression to be subtracted is changed and then added.

Example 4. *Add:* $13ab - 9cd - xy$ *and* $12cd - 4ab$.

Solution.

Column Method:

$$\begin{array}{r} 13ab - 9cd - xy \\ -4ab + 12cd \\ \hline 9ab + 3cd - xy \\ \hline \end{array}$$

Horizontal Method:

$(13ab - 9cd - xy) + (12cd - 4ab)$

$= (13ab - 4ab) + (-9cd + 12cd) - xy$ [Grouping like terms]

$= 9ab + 3cd - xy$

Example 5. *Subtract:* $x^3 - 3x - 1$ *from* $4x^3 - 2x^2 + 5$.

Solution.

Column Method:

$$\begin{array}{rrrr} 4x^3 & -\ 2x^2 & +\ 0x & +\ 5 \\ x^3 & +\ 0x^2 & -\ 3x & -\ 1 \\ - & - & + & + \\ \hline 3x^3 & -\ 2x^2 & +\ 3x & +\ 6 \\ \hline \end{array}$$

Horizontal Method:

$(4x^3 - 2x^2 + 5) - (x^3 - 3x - 1) = 4x^3 - 2x^2 + 5 - x^3 + 3x + 1$

$= 4x^3 - x^3 - 2x^2 + 3x + 5 + 1$

$= 3x^3 - 2x^2 + 3x + 6$

Multiplication of Algebraic Expressions

- Let us recall:

  The product of two numbers with like signs is positive and the product of two numbers with unlike signs is negative.

  For examples:

  (*i*) $(+4) \times (+5) = +(4 \times 5) = +20$ (*ii*) $(-6) \times (-2) = +(6 \times 2) = +12$

  (*iii*) $(+3) \times (-5) = -(3 \times 5) = -15$ (*iv*) $(-7) \times (+6) = -(7 \times 6) = -42$

- If x is a variable and a and b are positive integers, then

$$x^a \times x^b = x^{(a+b)}$$

For example:

$$x^5 \times x^3 = x^{(5+3)} = x^8$$

1. Multiplication of monomials

Product of monomials = (Product of their numerical coefficients) × (Product of their variable parts)

Examples 6. *Multiply* $2ab^2$ *by* $-3a^2b^3$.

Solution. Product of numerical coefficients = 2 × (–3) = –6

Product of their variable parts = $ab^2 \times a^2b^3 = a^3b^5$

The required product = $(-6) \times a^3b^5 = -6a^3b^5$

We can also directly write as

$$2ab^2 \times (-3a^2b^3) = (-3 \times 2)\,(ab^2 \times a^2b^3) = -6a^3b^5$$

2. Multiplication of a polynomial and a monomial

Multiply each term of the polynomial by the monomial using the distributive law.

Let a be a monomial and $(b + c)$ be a binomial. They can be multiplied as follows:

$$a \times (b + c) = (a \times b) + (a \times c)$$

Example 7. *Multiply* $(3x^2y - 8xy + 5y^2)$ *by* $(-2xy^2)$.

Solution. $(-2xy^2) \times (3x^2y - 8xy + 5y^2)$

$$= (-2xy^2) \times (3x^2y) + (-2xy^2) \times (-8xy) + (-2xy^2) \times (5y^2)$$

$$= -6x^3y^3 + 16x^2y^3 - 10xy^4$$

3. Multiplication of two binomials

Let $(a + b)$ and $(c + d)$ be two binomials. They can be multiplied as follows:

$$\begin{aligned}(a + b)(c + d) &= a \times (c + d) + b \times (c + d)\\ &= (a \times c + a \times d) + (b \times c + b \times d)\\ &= ac + ad + bc + bd\end{aligned}$$

Example 8. *Multiply* $(2a + 5b)$ *and* $(3a - 4b)$.

Solution.

$$\begin{aligned}(2a + 5b)(3a - 4b) &= 2a \times (3a - 4b) + 5b \times (3a - 4b)\\ &= (2a \times 3a - 2a \times 4b) + (5b \times 3a - 5b \times 4b)\\ &= 6a^2 - 8ab + 15ab - 20b^2\\ &= 6a^2 + 7ab - 20b^2\end{aligned}$$

Column method of multiplication:

$2a + 5b$		
$3a - 4b$		
$6a^2 + 15ab$		Multiplying by $3a$
$- 8ab - 20b^2$		Multiplying by $-4b$
$6a^2 + 7ab - 20b^2$		Adding the like terms

Note: The same procedure is applicable for algebraic expressions containing more than two terms.

Example 9. *Multiply* $(x^2 - 5x + 8)$ *and* $(x^2 + 2x - 3)$.

Solution.

$x^2 - 5x + 8$	
$\times\ x^2 + 2x - 3$	
$x^4 - 5x^3 + 8x^2$	Multiplying by x^2
$+ 2x^3 - 10x^2 + 16x$	Multiplying by $2x$
$- 3x^2 + 15x - 24$	Multiplying by -3
$x^4 - 3x^3 - 5x^2 + 31x - 24$	Adding the like terms

Division of Algebraic Expressions

In division too, the quotient of two numbers with like signs is positive and the quotient of two numbers with unlike signs is negative.

Recall if x is a variable and a and b are positive integers such that $a > b$, then $(x^a \div x^b) = x^{a-b}$

For example: $(x^7 \div x^3) = x^{7-3} = x^4$

1. Division of a monomial by a monomial

Division of a monomial by another monomial

= (Division of their numerical coefficients)

× (Division of their variable parts)

Example 10. *Divide* $6a^3b^2$ *by* $-2ab$.

Solution. $$\frac{6a^3b^2}{-2ab} = \left(\frac{6}{-2}\right) a^{(3-1)}\, b^{(2-1)} = -3a^2b$$

Example 11. *Divide* $-12a^4b^3c$ *by* $-4a^2b^2c$.

Solution. $$\frac{-12a^4b^3c}{-4a^2b^2c} = \left(\frac{-12}{-4}\right) a^{(4-2)} b^{(3-2)} c^{(1-1)}$$
$$= 3a^2bc^0 = 3a^2b \qquad [\because \quad c^0 = 1]$$

2. Division of a polynomial by a monomial

Each term of the polynomial is divided by the monomial.

Example 12. *Divide* $(15x^4y^2 + 12x^2y^2 - 9x^3y)$ *by* $(-3xy)$

Solution. $(15x^4y^2 + 12x^2y^2 - 9x^3y) \div (-3xy)$

$$= \frac{15x^4y^2}{-3xy} + \frac{12x^2y^2}{-3xy} - \frac{9x^3y}{-3xy}$$
$$= -5x^{4-1}y^{2-1} - 4x^{2-1}y^{2-1} + 3x^{3-1}y^{1-1}$$
$$= -5x^3y - 4xy + 3x^2 \qquad [\because \quad y^0 = 1]$$

3. Division of a polynomial by a polynomial

Example 13. *Divide* $12x^2 + 7xy - 12y^2$ *by* $3x + 4y$.

Solution.

Step 1: Arrange the terms of divisor and the dividend in descending order of their variable parts.

$$\begin{array}{r|l}
 & 4x - 3y \\
3x + 4y & 12x^2 + 7xy - 12y^2 \\
 & \underset{-}{+}12x^2 \underset{-}{+} 16xy \quad \text{[Changing signs]} \\
\hline
 & -9xy - 12y^2 \\
 & -9xy - 12y^2 \\
 & + \quad 0 + \quad 0 \quad \text{[Changing signs]} \\
\hline
 & 0
\end{array}$$

Step 2: Divide the first term of the dividend by the first term of the divisor to get the first term of the quotient, *i.e.*, $\frac{12x^2}{3x} = 4x$.

Step 3: Multiply all the terms of the divisor by the first term of the quotient and subtract the result from the dividend.

Step 4: Consider the remainder as the new dividend.

Step 5: Repeat the steps 2, 3 and 4 till the remainder obtained is zero or a polynomial of a degree less than that of the divisor.

Hence, $(12x^2 + 7xy - 12y^2) \div (3x + 4y) = 4x - 3y$.

EXERCISE 3.3

1. Find the value of algebraic expression by substituting the value of literals as given:

(*a*) $9xy + 4$ if = 2, $y = \frac{1}{3}$

(*b*) $a + b + c + a^2 + b^2$, if $a = 2$, $b = -3$, $c = 1$

(*c*) $x^2 - xy + y^2$, if $x = -2$, $y = 3$

2. Add the following algebraic expressions:

(*a*) $3x^2 - 5xy - 1 + 8y^2$; $-2xy + 3y^2 - 5 + 10x^2$; $8 - xy + x^2 - y^2$

(*b*) $2ax - 6by + 4cz$; $4by - 14ax$; $9cz - 4ax - 6by$

3. Subtract the following expressions:

(*a*) $4p^2 + 5q^2 + 7$ from $-4q^2 - 5r^2 - 6$

(*b*) $3x^2 - 6x + 4$ from $5 + x - 2x^2$

4. The perimeter of a triangle is $9y^2 - 9y + 4$ and its two sides are $2y^2 - 3y$ and $4y^2 + 6$. Find its third side.

5. Multiply:

(*a*) $(6a^2) \times (-4ab)$ (*b*) $\left(\frac{1}{3}xy\right) \times \left(-\frac{6}{7}x^2y\right)$

6. Multiply:

(*a*) $(-3x) \times (6x + 5)$ (*b*) $(5ab) \times (3a^2 - 4ab - b^2)$

7. Multiply:

(*a*) $(5a - 4) \times (7a + 5)$ (*b*) $(x^2 + x + 1) \times (1 - x)$

8. Multiply:

(*a*) $(2p^2 - 2pq + 3q^2) \times (3p^2 + 4pq + q^2)$

(*b*) $(x^3 - 5x^2 + 3x + 1) \times (x^2 - 3)$

9. Simplify:

(*a*) $3a(a + b) - 4b(a + b) + 2(ab + b^2)$

(*b*) $(x + y)(x + y + z) - (x - y)(x - y - z)$

10. Divide:

(*a*) $-18a^3b^2$ by $3ab$ (*b*) $75x^4y^2$ by $-25x^3y^2$

(*c*) $-20p^6q^5$ by $-4p^3q^2$

11. Divide:

(*a*) $12a^2b^4 + 16a^3b^3 - 20a^4b^2$ by $4a^2b^2$

(*b*) $8x^3yz - 24xy^3z + 48xyz^3$ by $(-8xyz)$

(*c*) $3abc - 6a^2bc + 4a^2b^2c^2$ by $(-3abc)$

12. Divide:

(*a*) $(x^4 + x^2y^2 + y^4)$ by $(x^2 - xy + y^2)$

(*b*) $(-8a^4 + 16a^3 - a + 2)$ by $(-2a^2 + 3a + 2)$

(*c*) $(16 + 8x + x^6 - 2x^4 + x^2)$ by $(x + 4 - x^3)$

3.5 EXPANSION

Consider the expression $2(x + 3)$. We say that 2 is the *coefficient* of the expression in the brackets. We can *expand* the brackets using the *distributive law*:

$$a(b + c) = ab + ac$$

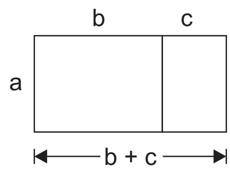
The distributive law says that we must multiply the coefficient by each term within the brackets, and add the results.

Geometric Demonstration:

The overall area is $a(b + c)$.

However, this could also be found by adding the areas of the two small rectangles: $ab + ac$.

So, $a(b + c) = ab + ac$. {equating areas}



Example 14. *Expand the following:*

(*a*) $3(4x + 1)$ (*b*) $2x(5 - 2x)$ (*c*) $x(2x - 1) - 2x(5 - x)$

Solution. (*a*) $3(4x + 1) = 3 \times 4x + 3 \times 1$

$= 12x + 3$

(*b*) $2x(5 - 2x) = 2x\,(5 - 2x)$

$= 2x \times 5 - 2x \times 2x = 10x - 4x^2$

(*c*) $x(2x - 1) - 2x(5 - x) = 2x^2 - x - 10x + 2x^2$

$= 4x^2 - 11x$

EXERCISE 3.4

1. Expand and simplify:

(*a*) $3(x + 1)$ (*b*) $2(5 - x)$ (*c*) $-(x + 2)$

(*d*) $-(3 - x)$ (*e*) $4(a + 2b)$ (*f*) $3(2x + y)$

2. Expand and simplify:

(*a*) $1 + 2(x + 2)$ (*b*) $13 - 4(x + 3)$ (*c*) $3(x - 2) + 5$

(*d*) $4(3 - x) - 10$ (*e*) $x(x - 1) + x$ (*f*) $2x(3 - x) + x^2$

3. Expand and simplify:

(*a*) $3(x - 4) + 2(5 + x)$ (*b*) $2a + (a - 2b)$

(*c*) $2a - (a - 2b)$ (*d*) $3(y + 1) + 6(2 - y)$

3.6 ALGEBRAIC FRACTIONS

Algebraic fractions are fractions that contain at least one variable either in numerator or in denominator or in both such as:

x is the numerator $\frac{x}{18}$	An expression in terms of x is the denominator $\frac{4}{x+2}$	The numerator is a multiple of x $\frac{2x}{15}$
Both the numerator and the denominator contains an x term $\frac{x+1}{2x}$	Both the numerator and the denominator contain an expression with x $\frac{3x+4}{2x-5}$	The numerator and the denominator are quadratic expressions $\frac{(3x+5)^2}{x^2-4}$

Adding and Subtracting Algebraic Fractions

When adding and subtracting fractions, we must ensure that we have the same denominator.

Example 15. *Calculate:* $\frac{2}{3} - \frac{y}{18}$.

Solution.
$$\frac{2}{3} - \frac{y}{18} = \frac{2 \times 18}{54} - \frac{3y}{54}$$
$$= \frac{36}{54} - \frac{3y}{54} = \frac{36 - 3y}{54} = \frac{3(12 - y)}{54} = \frac{12 - y}{18}.$$

Example 16. *Calculate:* $\frac{x}{y} + \frac{y}{x}$.

Solution.
$$\frac{x}{y} + \frac{y}{x} = \frac{x^2}{xy} + \frac{y^2}{xy} = \frac{x^2 + y^2}{xy}.$$

Example 17. *Simplify:* $\frac{2}{x} - \frac{5}{x+2}$.

Solution. $\frac{2}{x} - \frac{5}{x+2} = \frac{2(x+2)}{x(x+2)} - \frac{5x}{x(x+2)} = \frac{2x+4-5x}{x(x+2)} = \frac{4-3x}{x(x+2)}$.

EXERCISE 3.5

Simplify the following fractions. Write your answers in the lowest term.

1. $\frac{x}{3} + \frac{y}{3}$

2. $\frac{9}{8x} + \frac{6}{8x}$

3. $\frac{7}{8x} + \frac{5}{8x} - \frac{3}{8x}$

4. $\frac{12x-15}{12x} - \frac{9x-6}{12x}$

5. $\frac{ab}{4} + \frac{ab}{5}$

6. $\frac{7x}{8} + \frac{3x}{10} - \frac{x}{5}$

7. $\frac{2}{x^2} - \frac{3}{x^2} + \frac{7}{x}$

8. $\frac{x+y}{8} - \frac{x+y}{10}$

3.7 FACTORISATION

Factorisation is the process of writing an expression as a *product* of its *factors*.

Factorisation is the reverse process of expansion.

In *expansions* we have to *remove brackets*, whereas in *factorisation* we have to *insert brackets*. Notice that $3(x + 2)$ is the *product of two factors*, 3 and $x + 2$.

Expansion

$3(x + 2) = x + 6$

Factorisation

The brackets are essential, since, in $3(x + 2)$ the whole of $x + 2$ is multiplied by 3 whereas in $3x + 2$ only the x is multiplied by 3.

To factorise an algebraic expression involving a number of terms we look for the Highest Common Factor (HCF) of the terms and write it in front of a set of brackets. We then find the contents of the brackets.

For example: $5x^2$ and $10xy$ have HCF $5x$.

So, $5x^2 + 10xy = 5x \times x + 5x \times 2y$

$= 5x(x + 2y)$.

Factorise Fully

Notice that $4a + 12 = 2(2a + 6)$ is not fully factorised as $(2a + 6)$ still has a common factor of 2 which could be removed. Although 2 is a common factor it is not the *highest* common factor. The HCF is 4 and so

$4a + 12 = 4(a + 3)$ is fully factorised.

Example 18. *Fully factorise:*

(*a*) $3a + 6$ (*b*) $ab - 2bc$

Solution. (*a*) $3a + 6 = 3 \times a + 3 \times 2 = 3(a + 2)$ [HCF is 3]

(*b*) $ab - 2bc = a \times b - 2 \times b \times c = b(a - 2c)$ [HCF is b]

Example 19. *Fully factorise:*

(*a*) $8x^2 + 12x$ (*b*) $3y^2 - 6xy$

Solution. (*a*) $8x^2 + 12x = 2 \times 4 \times x \times x + 3 \times 4 \times x$

$= 4x(2x + 3)$ [HCF is $4x$]

(*b*) $3y^2 - 6xy = 3 \times y \times y - 2 \times 3 \times x \times y$

$= 3y(y - 2x)$ [HCF is $3y$]

Exmaple 20. *Fully factorise:*

(*a*) $2(x + 3) + x(x + 3)$ (*b*) $x(x + 4) - (x + 4)$

Solution. (*a*) $2(x + 3) + x(x + 3)$ [HCF is $(x + 3)$]

$= (x + 3)(2 + x)$

(*b*) $x(x + 4) - (x + 4) = x(x + 4) - 1(x + 4)$

$= (x + 4)(x - 1)$. [HCF is $(x + 4)$]

Example 21. *Fully factorise $(x - 1)(x + 2) + 3(x - 1)$.*

Solution. $(x - 1)[(x + 2) + 3]$ [HCF of $(x - 1)$]

$= (x - 1)[(x + 2) + 3]$

$= (x - 1)(x + 5)$.

EXERCISE 3.6

1. Copy and complete:

(*a*) $2x + 4 = 2(x + \ldots)$ (*b*) $3a - 12 = 3(a - \ldots)$

(*c*) $15 - 5p = 5(\ldots - p)$ (*d*) $18x + 12 = 6(\ldots + 2)$

2. Copy and complete:

(*a*) $4x + 16 = 4(\ldots + \ldots)$ (*b*) $10 + 5d = 5(\ldots + \ldots)$

(*c*) $5c - 5 = 5(\ldots - \ldots)$ (*d*) $cd + de = d(\ldots + \ldots)$

3. Fully factorise:

(*a*) $3a + 3b$ (*b*) $8x - 16$ (*c*) $3p + 18$ (*d*) $28 - 14x$

4. Fully factorise:

(*a*) $x^2 + 2x$ (*b*) $5x - 2x^2$ (*c*) $4x^2 + 8x$ (*d*) $14x - 7x^2$

5. Fully factorise:

(*a*) $-9a + 9b$ (*b*) $-3 + 6b$ (*c*) $-8a + 4b$ (*d*) $-7c + cd$

6. Fully factorise:

(*a*) $-6a - 6b$ (*b*) $-4 - 8x$ (*c*) $-3y - 6z$ (*d*) $-9c - cd$

7. Fully factorise:

(*a*) $2(x - 7) + x(x - 7)$ (*b*) $a(x + 3) + b(x + 3)$

(*c*) $4(x + 2) - x(x + 2)$ (*d*) $x(x + 9) + (x + 9)$

8. Fully factorise:

(*a*) $(x + 3)(x - 5) + 4(x + 3)$ (*b*) $5(x - 7) + (x - 7)(x + 2)$

(*c*) $(x + 6)(x + 4) - 8(x + 6)$ (*d*) $(x - 2)^2 - 6(x - 2)$

3.8 PRODUCT OF TWO BINOMIALS

ACTIVITY 1

Consider the figure alongside:

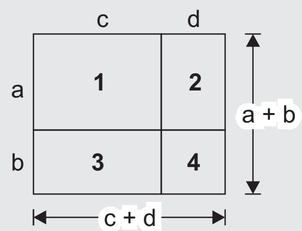
Give an expression for the area of:

(*a*) rectangle 1 (*b*) rectangle 2

(*c*) rectangle 3 (*d*) rectangle 4

(*e*) the overall rectangle.

What can you conclude?



The product (*a* + *b*) (*c* + *d*)

Consider the product $(a + b)(c + d)$.

It has two *factors,* $(a + b)$ and $(c + d)$.

We can evaluate this product by using the distributive law several times.

$$(a + b)(c + d) = a(c + d) + b(c + d)$$
$$= ac + ad + bc + bd$$

So, $(a + b)(c + d) = ac + ad + bc + bd$

Example 22. *Expand and simplify: (x + 3) (x + 2).*

Solution. $(x + 3)(x + 2) = x \times x + x \times 2 + 3 \times x + 3 \times 2$

$= x^2 + 2x + 3x + 6 = x^2 + 5x + 6.$

Example 23. *Expand and simplify:*

(*a*) $(x + 3)(x - 3)$ (*b*) $(3x - 5)(3x + 5)$.

Solution. (*a*) $(x + 3)(x - 3) = x^2 - 3x + 3x - 9 = x^2 - 9.$

(*b*) $(3x - 5)(3x + 5) = 9x^2 + 15x - 15x - 25 = 9x^2 - 25$

Example 24. *Expand and simplify:*

(*a*) $(3x + 1)^2$ (*b*) $(2x - 3)^2$

Solution. (*a*) $(3x + 1)^2 = (3x + 1)(3x + 1)$

$= 9x^2 + 3x + 3x + 1 = 9x^2 + 6x + 1$

(*b*) $(2x - 3)^2 = (2x - 3)(2x - 3)$

$= 4x^2 - 6x - 6x + 9 = 4x^2 - 12x + 9.$

EXERCISE 3.7

1. Expand and simplify:

(*a*) $(x + 3)(x + 7)$ (*b*) $(x + 5)(x - 4)$

(*c*) $(x - 3)(x + 6)$ (*d*) $(x + 2)(x - 2)$

2. Expand and simplify:

(*a*) $(x + 2)(x - 2)$ (*b*) $(a - 5)(a + 5)$

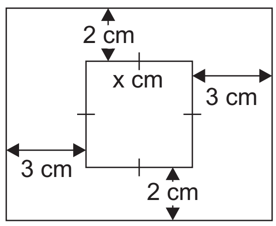
(*c*) $(4 + x)(4 - x)$ (*d*) $(2x + 1)(2x - 1)$

3. Expand and simplify:

(*a*) $(x + 3)^2$ (*b*) $(x - 2)^2$ (*c*) $(3x - 2)^2$ (*d*) $(1 - 3x)^2$

(*e*) $(3 - 4x)^2$ (*f*) $(5x - y)^2$

4. A square photograph has sides for length x cm. It is surrounded by a wooden frame with the dimensions shown. Show that the area of the rectangle formed by the outside of the frame is given by $A = (x^2 + 10x + 24)\ cm^2$.



3.9 DIFFERENCE OF TWO SQUARES

a^2 and b^2 are perfect squares and so $a^2 - b^2$ is called the *difference of two squares.* Notice that $(a + b)(a - b) = a^2 \underbrace{- ab + ab}_{\text{The middle two terms add to zero}} - b^2 = a^2 - b^2$

Thus, $(a + b)(a - b) = a^2 - b^2$

Geometric Demonstration:

Consider the figure alongside:

The shaded area

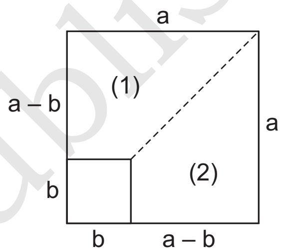
= area of large square – area of small square

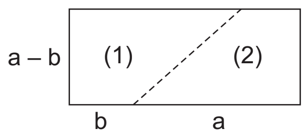
= $a^2 - b^2$

Cutting along the dotted line and flipping (2) over, we can form a rectangle.

The rectangle's area is $(a + b)(a - b)$.

$\therefore \quad (a + b)(a - b) = a^2 - b^2$.





Example 25. *Expand and simplify:*

(*a*) $(x + 5)(x - 5)$ (*b*) $(3 - y)(3 + y)$

(*c*) $(3x + 4y)(3x - 4y)$

Solution. (*a*) $(x + 5)(x - 5) = x^2 - 5^2 = x^2 - 25$

(*b*) $(3 - y)(3 + y) = 3^2 - y^2 = 9 - y^2$.

(*c*) $(3x + 4y)(3x - 4y) = (3x)^2 - (4y)^2 = 9x^2 - 16y^2$.

3.10 PERFECT SQUARE EXPANSION

ACTIVITY 2

Consider the figure alongside:

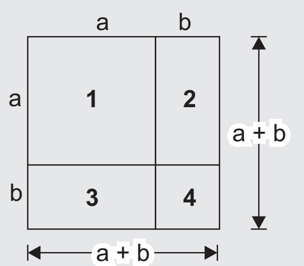
Give an expression for the area of:

(*a*) square 1 (*b*) rectangle 2

(*c*) rectangle 3 (*d*) rectangle 4

(*e*) the overall rectangle.

What can you conclude?



$(a + b)^2$ and $(a - b)^2$ are called *perfect squares.*

Notice that $$(a + b)^2 = (a + b)\,(a + b)$$
$$= a^2 + ab + ab + b^2$$
$$= a^2 + 2ab + b^2$$

Thus, we can state the perfect square expansion rule:

$$(a + b)^2 = a^2 + 2ab + b^2$$

We can remember the rule as follows:

Step 1: Square the *first term.*

Step 2: Add twice the product of the *first* and *last terms.*

Step 3: Add on the square of the *last term.*

Notice that $$(a - b)^2 = (a + (-b))^2$$
$$= a^2 + 2a(-b) + (-b)^2$$
$$= a^2 - 2ab - b^2$$

Once again, we have the square of the first term, twice the product of the first and last terms, and the square of the last term.

Example 26. *Expand and simplify:*

(*a*) $(x + 3)^2$ (*b*) $(x - 5)^2$

Solution. (*a*) $(x + 3)^2 = x^2 + 2 \times x \times 3 + 3^2 = x^2 + 6x + 9$

(*b*) $(x - 5)^2 = (x - 5)^2$
$$= x^2 + 2 \times x \times (-5) + (-5)^2 = x^2 - 10x + 25.$$

Example 27. *Expand and simplify:*

(*a*) $(2x^2 + 3)^2$ (*b*) $5 - (x + 2)^2$

Solution. (*a*) $(2x^2 + 3)^2 = (2x^2)^2 + 2 \times 2x^2 \times 3 + 3^2$
$$= 4x^4 + 12x^2 + 9$$

(*b*) $5 - (x + 2)^2 = 5 - [x^2 + 4x + 4]$
$$= 5 - x^2 - 4x - 4 = 1 - x^2 - 4x.$$

EXERCISE 3.8

1. Expand and simplify using the rule $(a + b)(a - b) = a^2 - b^2$:

(*a*) $(x + 2)\,(x - 2)$ (*b*) $(x - 3)\,(x + 3)$

2. Expand and simplify using the rule $(a + b)\,(a - b) = a^2 - b^2$:

(*a*) $(2x - 1)\,(2x + 1)$ (*b*) $(3x + 2)\,(3x - 2)$

(*c*) $(4y - 5)\,(4y + 5)$ (*d*) $(2y + 5)\,(2y - 5)$

3. Expand and simplify using the rule $(a + b)(a - b) = a^2 - b^2$:

(*a*) $(2a + b)(2a - b)$ (*b*) $(a - 2b)(a + 2b)$

(*c*) $(4x + y)(4x - y)$ (*d*) $(4x + 5y)(4x - 5y)$

4. (*a*) Using the difference of two squares expansion to show that:

(*i*) $43 \times 37 = 40^2 - 3^2$ (*ii*) $24 \times 26 = 25^2 - 1^2$

(*b*) Evaluate without using a calculator:

(*i*) 18×22 (*ii*) 49×51 (*iii*) 103×97.

5. Use the rule $(a + b)^2 = a^2 + 2ab + b^2$ to expand and simplify:

(*a*) $(x + 5)^2$ (*b*) $(x + 4)^2$ (*c*) $(x + 7)^2$

6. Expand and simplify:

(*a*) $(x - 3)^2$ (*b*) $(x - 2)^2$ (*c*) $(y - 8)^2$

7. Expand and simplify:

(*a*) $(3x + 4)^2$ (*b*) $(2a - 3)^2$ (*c*) $(3y + 1)^2$

8. Expand and simplify:

(*a*) $(x^2 + 2)^2$ (*b*) $(y^2 - 3)^2$ (*c*) $(3a^2 + 4)^2$

9. Expand and simplify:

(*a*) $3x + 1 - (x + 3)^2$ (*b*) $5x - 2 + (x - 2)^2$

(*c*) $(x + 2)(x - 2) + (x + 3)^2$ (*d*) $(x + 2)(x - 2) - (x + 3)^2$

3.11 FACTORISATION OF QUADRATIC EXPRESSIONS

Splitting the Middle Term

A *quadratic expression* is a quadratic trinomial of the form $ax^2 + bx + c$, where x is a variable and a, b, c are constants, $a \neq 0$.

Here, we will learn a useful technique for factorisation of a quadratic expression by *splitting the middle term.*

Consider $(2x + 3)(4x + 5)$

$$= 8x^2 + 10x + 12x + 15$$
$$= 8x^2 + 22x + 15$$

In reverse, $8x^2 + 22x + 15$

$$= \underbrace{8x^2 + 10x} + \underbrace{12x + 15}$$
$$= 2x(4x + 5) + 3(4x + 5)$$
$$= (4x + 5)(2x + 3)$$

So, we can factorise $8x^2 + 22x + 15$ into $(2x + 3)(4x + 5)$ by splitting the $+ 22x$ into a suitable sum, in this case $+ 10x + 12x$.

In general, if we start with a quadratic trinomial we will need a method to work out how to do the splitting.

Consider the expension in general detail:

$$(2x + 3)(4x + 5) = 2 \times 4 \times x^2 + [2 \times 5 + 3 \times 4]x + 3 \times 5$$
$$= 8x^2 + 22x + 15$$

The four numbers 2, 3, 4 and 5 are present in the *middle term,* and also in the *first* and *last* terms combined.

As 2 × 5 and 3 × 4 are factors of 2 × 3 × 4 × 5 = 120 this gives us the method for performing the splitting.

Step 1: Multiply the coefficient of x^2 and the constant term.

In our case, 8 × 15 = 120

Step 2: Look for the factors of this number which add to give the coefficient of the middle term.

What factors of 120 add to give us 22? The answer is 10 and 12.

Step 3: These numbers are the coefficients of the split terms.

So, the split is $10x + 12x$.

Example 28. *Factorise* $3x^2 + 17x + 10$.

Solution. For $3x^2 + 17x + 10$, 3 × 10 = 30

We need to find two factors of 30 which have a sum of 17. These are 2 and 15.

$$\therefore \quad 3x^2 + 17x + 10 = 3x^2 + 2x + 15x + 10$$
$$= x(3x + 2) + 5(3x + 2)$$
$$= (3x + 2)(x + 5).$$

Example 29. *Factorise:* $x^2 - 7x + 12$.

Solution. For $x^2 - 7x + 12$, 1 × 12 = 12

$$= x^2 - 4x - 3x + 12$$
$$= x(x - 4) - 3(x - 4)$$

We need to find two factors of 12 which have a sum – 7. These are – 3 and – 4.

$$\therefore \quad x^2 - 7x + 12 = (x - 3)(x - 4)$$

Remark. The sum is negative but the product is positive, so both numbers must be negative.

Example 30. *Factorise: $6x^2 - 11x - 10$.*

Solution. $6x^2 - 11x - 10$, $6 \times -10 = -60$

We need to find two factors of – 60 which have a sum of – 11.

These are – 15 and 4.

$$\therefore \quad 6x^2 - 11x - 10 = 6x^2 - 15x + 4x - 10$$
$$= 3x(2x - 5) + 2(2x - 5)$$
$$= (2x - 5)(3x + 2).$$

Remark. As the product is negative, the two numbers must be opposite in sign. Since the sum is negative, the larger number must be negative.

EXERCISE 3.9

1. Factorise:

(*a*) $x^2 + 4x + 3$ (*b*) $x^2 + 14x + 24$

(*c*) $x^2 + 10x + 21$ (*d*) $x^2 + 15x + 54$

2. Factorise:

(*a*) $x^2 - 3x + 2$ (*b*) $x^2 - 4x + 3$

(*c*) $x^2 - 5x + 6$ (*d*) $x^2 - 14x + 33$

3. Fully factorise:

(*a*) $2x^2 + 5x + 3$ (*b*) $2x^2 + 7x + 5$

(*c*) $7x^2 + 9x + 2$ (*d*) $3x^2 + 7x + 4$

4. Fully factorise:

(*a*) $2x^2 - 9x - 5$ (*b*) $3x^2 + 5x - 2$

(*c*) $3x^2 - 5x - 2$ (*d*) $2x^2 + 3x - 2$

5. Fully factorise:

(*a*) $15x^2 + 19x + 6$ (*b*) $15x^2 + x - 6$

(*c*) $15x^2 - x - 6$ (*d*) $30x^2 - 38x + 12$

REVIEW EXERCISE

1. Form algebraic expressions of the following statements.

(*a*) 10 more than twice a number.

(*b*) Product of x and y added to z.

(*c*) Difference of x and y divided by 5.

(*d*) Product of x and y added to their difference.

(*e*) Twice the number x added to square of it.

(*f*) Add 9 to the number x and divide the sum by 3.

(*g*) Subtract the number x from product of y and 3.

(*h*) Divide the difference of x and y by 2.

2. Find the value of algebraic expression by substituting the value of literals as given:

(*a*) $p^2q^2r^2 + 7$, if $p = 4$, $q = -1$, $r = 2$

(*b*) $x + y + z + xyz$, if $x = 9$, $y = -3$, $z = -1$

3. Add the algebraic expressions: $2x^3 - 9x^2 + 8$; $3x^2 - 6x - 5$; $7x^3 + 10x + 1$

4. Subtract the following expressions: $a^2 + ab + b^2$ from $4a^2 - 3ab + 2b^2$.

5. Multiply: $(-3ab^2) \times (3ab^2) \times (-2a^2b)$

6. Multiply: $\left(-\frac{1}{2}ab^2\right) \times (2a^2 - 3b^2)$

7. Multiply: $\left(3x + \frac{1}{2}y\right) \times \left(3x - \frac{1}{2}y\right)$

8. Multiply: $(a^3 - a^2b^2 + b^3) \times (a^2 - b^2)$

9. Simplify: $x(x + 4) + 3x(2x^2 - 1) + 4x^2 + 4$

10. Find the quotient and the remainder in the division of:

(*a*) $(a^3 - 5a^2 + 8a + 15)$ by $(a + 1)$

(*b*) $(5x^3 - 4c^2 + 2x - 3)$ by $(x^2 + 2x - 1)$

(*c*) $(a^4 + a^2 + 2)$ by $(a^2 - a + 1)$

11. Expand and simplify:

(*a*) $5(x - y)$ (*b*) $6(-x^2 + y^2)$ (*c*) $-2(x + 4)$

(*d*) $-3(2x - 1)$ (*e*) $x(x + 3)$ (*f*) $2x(x - 5)$

12. Expand and simplify:

(*a*) $2a(b - a) + 3a^2$ (*b*) $4x - 3x(x - 1)$ (*c*) $7x^2 - 5x(x + 2)$

13. Expand and simplify:

(*a*) $2(y - 3) - 4(2y + 1)$ (*b*) $3x - 4(2 - 3x)$

(*c*) $2(b - a) + 3(a + b)$ (*d*) $x(x + 4) + 2(x - 3)$

Simplify the following fractions. Write your answers in the lowest term.

14. $\frac{x}{a^2b} + \frac{y}{ab^2}$

15. $\frac{x + 7}{3} + \frac{2x - 3}{5}$

16. $\frac{2x}{x + 1} + \frac{3}{(x + 1)^2}$

17. $\frac{3}{(x + 1)} - \frac{6}{(x + 1)(x + 3)}$

18. Copy and complete:

(*a*) $4x^2 - 8x = 4x(x - \ldots)$ (*b*) $2m + 8m^2 = 2m(\ldots + 4m)$

19. Copy and complete:

(*a*) $6a + 8ab = \ldots(3 + 4b)$ (*b*) $6x - 2x^2 = \ldots(3 - x)$

(*c*) $7ab - 7a = \ldots (b - 1)$ (*d*) $4ab - 6bc = \ldots(2a - 3c)$

20. Expand and simplify:

(*a*) $(x - 8)(x + 3)$ (*b*) $(2x + 1)(3x + 4)$

(*c*) $(1 - 2x)(4x + 1)$ (*d*) $(4 - x)(2x + 3)$

21. Expand and simplify:

(*a*) $(5a + 3)(5a - 3)$ (*b*) $(4 + 3a)(4 - 3a)$

22. Expand and simplify using the rule $(a + b)(a - b) = a^2 - b^2$:

(*a*) $(x + 1)(x - 1)$ (*b*) $(1 - x)(1 + x)$

23. Expand and simplify using the rule $(a + b)(a - b) = a^2 - b^2$:

(*a*) $(3x + 1)(3x - 1)$ (*b*) $(1 - 3x)(1 + 3x)$

24. Expand and simplify using the rule $(a + b)(a - b) = a^2 - b^2$:

(*a*) $(2x + 3y)(2x - 3y)$ (*b*) $(7x - 2y)(7x + 2y)$

25. Expand and simplify:

(*a*) $(3 - 2x)^2 - (x - 1)(x + 2)$ (*b*) $(1 - 3x)^2 + (x + 2)(x - 3)$

26. Factorise:

(*a*) $x^2 + 9x + 20$ (*b*) $x^2 + 8x + 15$

27. Factorise:

(*a*) $x^2 - 16x + 39$ (*b*) $x^2 - 19x + 48$

28. Fully factorise:

(*a*) $3x^2 + 13x + 4$ (*b*) $3x^2 + 8x + 4$

29. Fully factorise:

(*a*) $2x^2 + 3x - 5$ (*b*) $5x^2 - 14x - 3$

30. Fully factorise:

(*a*) $18x^2 - 12x + 2$ (*b*) $48x^2 + 72x + 27$

MULTIPLE CHOICE QUESTIONS (MCQs)

1. In the term $(-2pqr)$, numerical coefficient of p is:

(*a*) $-2qr$ (*b*) 2 (*c*) -2 (*d*) $2qr$

2. $14x^2y^2z$ and $7x^2zy^2$ are:

(*a*) unlike terms (*b*) like terms (*c*) binomial (*d*) None of these

3. If the length of a line is $2a + 3b$, if it is shortened by the measure $(a + b)$, find the remaining length of the line

(*a*) $3a + 4b$ (*b*) $a + 2b$ (*c*) $a - 2b$ (*d*) $a + b$

4. Daniel went to a park at a distance $x^2 + 2x + 3$ and came back on the same path. Find the total distance he travelled.

(*a*) $x^2 + 2x + 3$ (*b*) $2x^2 + 4x + 6$ (*c*) $2x^2 + 4x + 3$ (*d*) None of these

5. One side of a square plot is $(a + b + c)$. Find the perimeter.

(*a*) $2a + 2b + 2c$ (*b*) $4(a + b + c)$ (*c*) $4a + b + c$ (*d*) None of these

6. If we add $x^2 + 8x - 3$ and $-x^2 + 3 - 8x$, we get

(*a*) $2x^2 + 16x + 6$ (*b*) $-2x^2 - 16x - 6$ (*c*) 0 (*d*) 6

7. If $x = 3$, $y = -4$, $z = 2$, find the value of $x + y + z$.

(*a*) 9 (*b*) –9 (*c*) –1 (*d*) +1

8. Subtract the sum of $x^2 + 7x + 3$ and $3x^2 + 2x + 2$ from 1.

(*a*) $4x^2 + 9x + 4$ (*b*) $-4x^2 - 9x - 4$ (*c*) $-4x^2 - 9x - 5$ (*d*) None of these

9. $(x + 2)(x + 2) = ?$

(*a*) $(x^2 + 2x + 4)$ (*b*) $(x^2 + 4x + 4)$ (*c*) $(x^2 + 4)$ (*d*) $x^2 - 4$

10. $\left(\frac{1}{a}+\frac{1}{b}\right)\left(\frac{1}{a}-\frac{1}{b}\right) = ?$

(*a*) $\left(\frac{1}{a^2}-\frac{1}{b^2}\right)$ (*b*) $\left(\frac{1}{a^2}+\frac{1}{b^2}\right)$ (*c*) $\left(\frac{1}{a}+\frac{1}{b}-\frac{1}{b}\right)$ (*d*) $\frac{a^2-b^2}{ab}$

RECAP AT A GLANCE

- In algebra, a statement is basically sentences which are *either* true *or* false.
- A numerical statement is a *mathematical phrase* or *problem made up entirely of numerical characters.*
- In order to evaluate an algebraic expression, you must know the exact values for each variable.
- The product of two numbers with like signs is positive and the product of two numbers with unlike signs is negative.
- The quotient of two numbers with like signs is positive and the quotient of two numbers with unlike signs is negative.
- Algebraic fractions are fractions that contain at least one variable either in numerator or in denominator or in both.
- *Factorisation* is the process of writing an expression as a *product* of its *factors.*
- *Factorisation* is the reverse process of expansion.
- A *quadratic expression* is a quadratic trinomial of the form $ax^2 + bx + c$, where x is a variable and a, b, c are constants, $a \neq 0$.

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